A note on the triangle-centered quadratic interpolation discretization of the shape operator

J . Reisman (NYU), E. Grinspun (Columbia), D. Zorin (NYU)

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Abstract

In this note we consider a simple shape operator discretization for general meshes, based on computing an interpolating quadratic function passing through vertices of a triangle and its edge-adjacent neighbors. This approximation is computationally simple and consistent for a broad class of meshes. However, its convergence properties in the context of mesh optimization problems are not as good as some of the previously proposed techniques and it suffers from instabilities for certain point configurations.

In [GGRZ06], we explored the behavior of a number of discretizations of the shape operator on general meshes. Other than the midedge normal discrete operator introduced in that paper, all these operators were using vertex degrees of freedom (DOFs) only, and were obtained one of three minimal stencils: *edge-centered* (vertices of two triangles sharing an edge), *face-centered* (vertices of all triangles adjacent to a given triangle), and *vertex-centered* (vertices of all triangles sharing a given vertex). The stencils are shown in Figure 1.

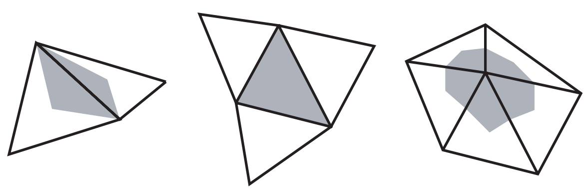

Figure 1: Minimal stencils: edge-centered, triangle-centered and vertex-centered.

Once a stencil is fixed, there are two common approaches to defining a shape operator or mean curvature vector: interpolating or approximating quadratic approximation to the surface, and elementary hinge operator averaging. The latter approach is motivated by discrete geometric ideas, i.e., defining shape operators using properties not requiring surface smoothness (see [CSM03]). In this case, the shape operator associated with an edge is approximated by an operator with one principal curvature direction aligned with the edge, and the only non-zero principal curvature magnitude proportional to the angle between triangle normals.

In the case of the simplest stencil (two triangles sharing an edge) the two approaches are effectively the same, as only a cylinder can be used to interpolate four points. One type of quadratic fit [Tau95] and discrete hinge averaging [PP93] were considered for vertex-centered stencils in [GGRZ06]. However, [GGRZ06] presents the results only the hinge-averaging operator for the triangle-centered stencil. Quadratic fit can be also applied on the triangle-centered stencil: the number of DOFs for triangles with no vertices of valence 3, exactly matches the number of DOFs needed for a general quadratic function, so a quadratic fit yields an interpolating quadratic function. As it was pointed out in [Zor05] this is sufficient for consistency of discretization, but is in general insufficient for convergence.

This note presents the results that were obtained for this operator at the time [GGRZ06] was written, but were omitted from the experimental results due to space limitations and the lack of observed advantages of this operator compared to other formulations. Additionally, we present explicit formulas for the operator obtained using elementary geometry, rather than solving a linear system of equations.

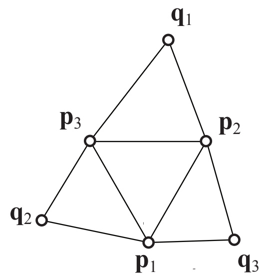

Figure 2: Notation for stencil vertices.

Formulation. The idea is to approximate the surface locally using a quadratic function defined over a plane close to the tangent plane. We use the plane of the central triangle of the stencil as the tangent plane approximation.

Let the triangle normal be $\mathbf{n}$. The distances from points $\mathbf{q}_i$, $i = 1 \ldots 3$ (see Figure 2) to the plane of the triangle are computed as $(\mathbf{q}_i - \mathbf{p}_i) \cdot \mathbf{n} = f(q_i)$. Let $\tilde{\mathbf{q}}_i = \mathbf{q}_i - f(q_i)\mathbf{n}_i$ be projections of points $\mathbf{q}_i$ to the plane of the triangle.

We seek a quadratic function Q defined on the plane of the triangle, satisfying $Q(\mathbf{p}_i) = 0$ (interpolation of points) and $Q(\tilde{\mathbf{q}}_i) = f(q_i)$, $i = 1 \ldots 3$, and use its quadratic term coefficients to estimate the Hessian. Such quadratic function may be nonunique (if some of the points coincide) or not exist (if six points $\mathbf{p}_i$, $\tilde{\mathbf{q}}_i$ are on the same conic). Whenever six points of the stencil are *close* to a common conic, the coefficients of the quadratic interpolant become large and Hessian estimation becomes highly unreliable.

We define $\mathbf{w}_i = \mathbf{p}_{i+1}$ and $\mathbf{w}_{i+3} = \tilde{\mathbf{q}}_i$, $i = 0 \ldots 2$. Let z_i be $f(\mathbf{w}_i)$, $i = 0 \ldots 5$.

Proposition 1. *For six co-planar points $\mathbf{w}_i$ $i = 0 \ldots 5$ in a plane P, there is a unique quadratic function on P satisfying $Q(\mathbf{w}_i) = z_i$, for arbitrary choice of z_i, if and only if these six points are not on the same conic.*

Proof. Assume that the origin is not in the plane P (if the plane passes through the origin, it can be shifted away from origin along its normal changing the quadratic function only by a constant). We use three-dimensional representations of points $\mathbf{w}_i$ in P. If A is a rotation mapping P to a plane $z = C$, then $\bar{\mathbf{x}} = A\mathbf{x}$ for any point $\mathbf{x}$ in the plane is of the form $[x, y, C]$, i.e., is a homogeneous form of the points. It is well-known that any quadratic function in homogeneous coordinates can be written as $\bar{\mathbf{x}}^T \bar{Q} \bar{\mathbf{x}}$. Then we obtain the matrix Q in the original coordinate system as $A^T \bar{Q} A$, and $Q(\mathbf{w}_i) = \mathbf{w}_i^T Q \mathbf{w}_i$.

By linearity of Q with respect to z_i, Q can be expressed as a linear combination of six basis matrices Q_j, $j = 0..5$, where Q_j is the quadratic form for the configuration $z_i = \delta_{ij}$.

We now derive the expression for Q_0, i.e., for the configuration $z_i = 0$ for $i \geq 1$, and $z_i = 0$; expressions for other Q_j are obtained by circular permutation of points. Note that in this case all points except $\mathbf{w}_0$ are in the same plane. As $z_i = 0$ for $i \geq 1$ the quadratic function Q vanishes at $\mathbf{w}_i$, for any $i \geq 1$: $\mathbf{w}_i^T Q_0 \mathbf{w}_i = 0$, i.e., the matrix Q_0 defines a conic passing through five points. Such a conic is unique, unless three points are collinear. (We consider the question of uniqueness below). Conversely, a conic passing through five points $\mathbf{w}_i$, $i \geq 1$, determines Q_0 up to a constant. Given a nontrivial conic with matrix M passing through $\mathbf{w}_i$, $i \geq 1$, we obtain Q_0 as $M/(\mathbf{w}_0^T M \mathbf{w}_0)$. If $\mathbf{w}_0^T M \mathbf{w}_0 = 0$, either there are multiple conics passing through the points, or the system has no solution.

□

A conic matrix M can be computed using a matrix form of the Braikenridge-Maclaurin construction, [CG67] (Figure 3) which we review here for completeness.

This construction is based on Pascal's theorem. Construct three pairs of lines, each pair passing through opposing sides of the hexagon $(\mathbf{x}, \mathbf{w}_1, \ldots \mathbf{w}_5)$. Each pair of lines will intersect at some (possibly infinite) point. By Pascal's theorem, x lies on the conic passing through $\mathbf{w}_i$ if and only if all three intersection points

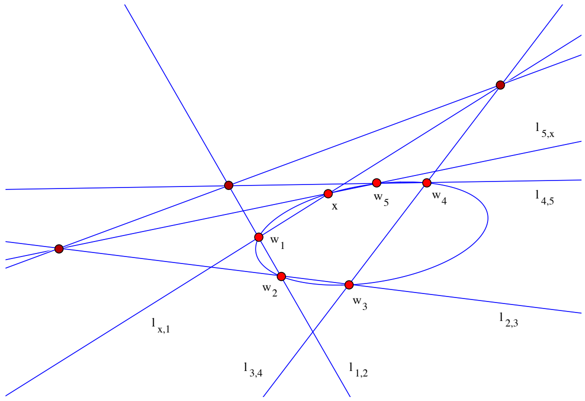

Figure 3: Constructing a conic passing through 5 points $\mathbf{w}_i$, $i = 0 \dots 6$.

are collinear. For two points $\mathbf{a}$ and $\mathbf{b}$, on a plane P not containing the origin, the line $(\mathbf{a}, \mathbf{b})$ passing through these points can be identified with $\mathbf{a} \times \mathbf{b}$: this vector determines a plane P', passing through the origin such that $P \cap P' = (\mathbf{a}, \mathbf{b})$. Similarly, for two intersecting lines l_a and l_b in P, corresponding to vectors $\mathbf{a}$ and $\mathbf{b}$, $\mathbf{a} \times \mathbf{b}$ defines a line through the origin passing through $l_a \cap l_b$. This also works for two distinct parallel lines (in this case the intersection point is at infinity).

Using these formulas we write Pascal's theorem algebraically. Define $\mathbf{l}_{i,i+1} = \mathbf{w}_i \times \mathbf{w}_{i+1}$, the lines along the edges of the hexagon not containing $\mathbf{x}$. The intersection points of opposite sides are given by $\mathbf{r}_1 = (\mathbf{w}_1 \times \mathbf{x}) \times \mathbf{l}_{34}$, $\mathbf{r}_2 = \mathbf{l}_{12} \times \mathbf{l}_{45}$, and $\mathbf{r}_3 = \mathbf{l}_{23} \times (\mathbf{w}_5 \times x)$. (Here we rely on the fact that no three points are collinear, so no two lines coincide).

Using this notation, Pascal's theorem's statement, becomes

$$\mathbf{r}_1 \cdot (\mathbf{r}_2 \times \mathbf{r}_3) = 0,$$

i.e., the volume of the parallelepiped spanned by $\mathbf{r}_i$ is zero. In expanded form, this yields a quadratic equation for $\mathbf{x}$:

$$(\mathbf{l}_{34} \times (\mathbf{w}_1 \times \mathbf{x})) \cdot (\mathbf{r}_2 \times (\mathbf{l}_{23} \times (\mathbf{w}_5 \times \mathbf{x}))) = 0$$

This formula allows us to write an explicit expression for the matrix M.Let $R(\mathbf{a})$ be the skew symmetric matrix satisfying $R\mathbf{x} = \mathbf{a} \times \mathbf{x}$. Replacing $\mathbf{a} \times \cdot$ with left multiplication by $R(\mathbf{a})$ in the formula above, we obtain

$$(R(\mathbf{l}_{34})R(\mathbf{w}_1)\mathbf{x})) \cdot (R(\mathbf{r}_2)R(\mathbf{l}_{23})R(\mathbf{w}_5)\mathbf{x}) = \mathbf{x}^T R(\mathbf{w}_1)R(\mathbf{l}_{34})R(\mathbf{r}_2)R(\mathbf{l}_{23})R(\mathbf{w}_5)\mathbf{x} = 0$$

(we use skew-symmetry of $R(\mathbf{a})$: $R(\mathbf{a})^T = -R(\mathbf{a})$).

This yields an expression for M:

$$M = R(\mathbf{w}_1)R(\mathbf{l}_{34})R(\mathbf{r})R(\mathbf{l}_{23})R(\mathbf{w}_5),$$

A complete discretization of the shape operator is given by

$$S_{quad} = \sum_{i=3}^{6} Q_i f(\tilde{\mathbf{q}}_i) = \sum_{i=3}^{6} Q_i(\mathbf{n} \cdot (\mathbf{q}_i - \mathbf{p}_i))$$

Only the last 3 matrices Q_i corresponding to $\tilde{\mathbf{q}}_i$ are used, as the values of the quadratic function at $\mathbf{p}_i$, $i = 1 \dots 3$ is always zero.

Uniqueness. The quadratic function exists and is unique if six points $\mathbf{w}_i$ are not collinear and are not on the same conic (possibly degenerate).

Suppose some three points are collinear but no four points are collinear. In this case, when we construct matrices Q_i according to the procedure above, the quadratic function is still uniquely defined: For each Q_i, we may have three out of five defining points on one line, but the remaining two are not on the same line. The collinear triple and the remaining pair define two lines, i.e., a degenerate conic defined by the product of two linear equations.

If four or five points are collinear, it is easy to see that the quadratic function is not defined uniquely. In this case one can pick a unique degenerate conic passing through four collinear and one additional point by requiring it to be two parallel lines; we note that for such configurations resulting conics are unstable with respect to small perturbations of points.

Boundaries and valence 3 vertices. For boundaries, as one of the flap triangles of the stencil may be missing, we use reflection about the boundary edge to create an additional point. If one of the triangle vertices has valence three, the stencil centered at this triangle has five or four points. If there are only four points, this implies that the whole mesh is a tetrahedron, thus only the case of five points is interesting. In this case, we use an additional vertex outside the stencil for the quadratic fit.

Note that if all six points of the stencil projected to the central triangle plane are close to a conic, the quadratic function still exists and is unique but does not yield a good curvature estimate: this is a fundamental limitation of the discretization. In the case when six distinct points are exactly on a unique conic (e.g., vertices of a regular hexahedron), there is no interpolating quadratic function.

Evaluation. We tested this discretization in three convergence experiments, identical to the ones described in [GGRZ06]. Two tests are based on the linearized form of the operator used to discretize thin-plate energy: uniformly loaded plate deformation and recovery of a quadratic displacements in the interior of the given Dirichlet and Neumann boundary conditions sampled from the quadratic displacement function (Neumann conditions are enforced by sampling two rows of points along the boundary). The third test uses the complete (nonlinear in vertex displacements) form of the operator to discretize the Willmore energy; Dirichlet and Neumann boundary conditions sampled from the sphere, which minimizes the Willmore energy in the continuum case.

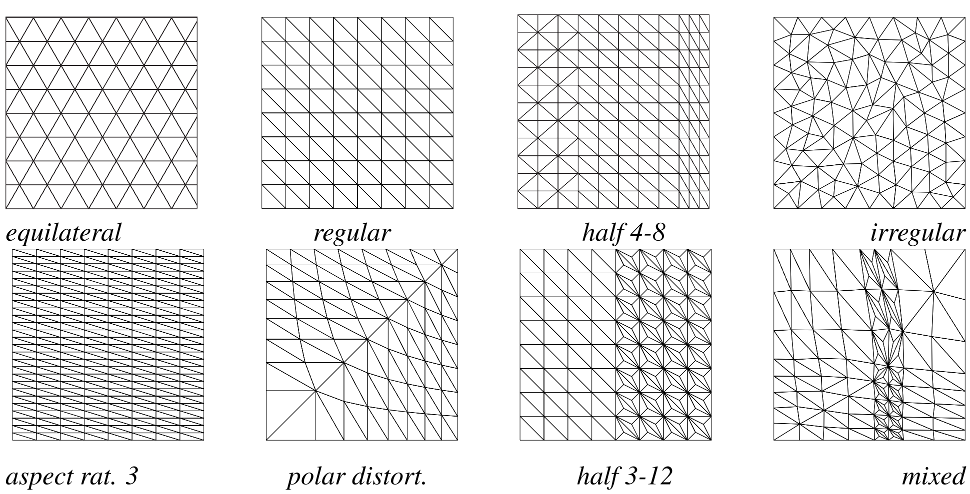

Figure 4: Mesh types.

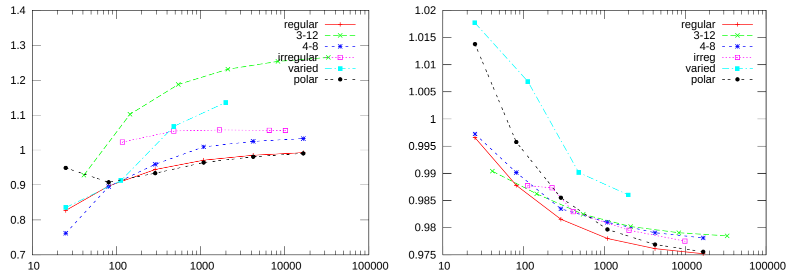

Figure 5: Left: uniformly loaded linearized plate test for different mesh types. Right: Sphere recovery test. The maximal computed pointwise displacement is shown relative to the maximal pointwise displacement for the analytic solution.

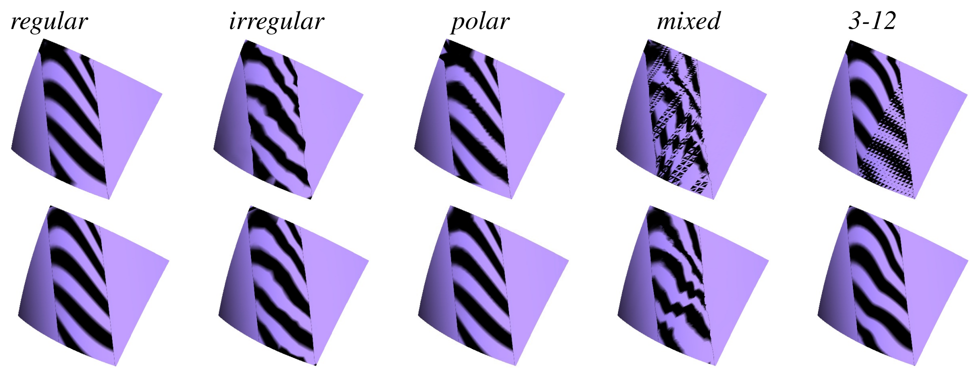

Figure 6: The boundary conditions (two rows of vertices) are sampled from a quadratic function. Linear thin-plate functional is minimized. To compute normals at vertices the top row averages the normals of triangles incident to each vertex, while the bottom row fits a quadratic to the area of the surface surrounding each vertex and uses the corresponding normal.

We used several types of meshes shown in Figure 4. The results for different tests are shown in are shown in Figures 5 and 6.

Comparing to operators used in [GGRZ06], we observe that the quality of the results is somewhat better than triangle-averaged and vertex quadratic fit discretizations, but inferior to the cotangent formula discretization and midedge normal discretization.

Note that [GGRZ06] uses a fit to edge-wise normal curvature approximations [MS92, Tau95, SK01], rather than a quadratic fit to the ring of vertices as it was done in [WW94], with special treatment for valence three and four vertices and degenerate cases. The quality relative to the latter type of fit is unknown to us.

Figure 6 shows that the the normal computation has a significant impact on visual quality. For most meshes, relatively expensive techniques based on fitting significantly improve the appearance, unless the normals are computed as a part of the optimization process as it is done for the midedge operator.

We observe that the quality of the results for tests with no external forces is better than that with external forces applied (compare quadratic surface recovery errors shown in Figure 7 to the plate and sphere recovery errors in Figure 5). However, the cotangent formula appears to yield somewhat better results, and its computational cost is comparable, if not better. We note that cotangent discretization does not yield a full shape operator, only the mean curvature (for generalization to a shape operator discretization, see [HPW05]).

Considering triangle-centered stencils, quadratic interpolation is less robust, and more complex, than hinge averaging. In particular, (i) near-conic configurations of stencil vertices lead to numerical instabilities, (ii) vertices of valence three require special treatment, and (iii) the formulas for the discretization coefficients

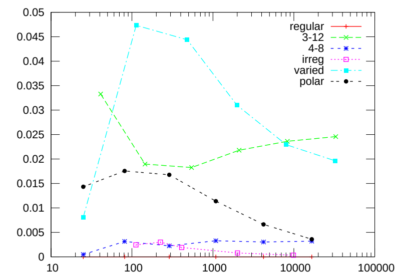

Figure 7: Error plots for quadratic function recovery for different types. The maximal error is measured relative to difference between maximal and minimal values (vertical extent) of the solution.

are more costly than corresponding formulas for hinge averaging, an important consideration for nonlinear problems with significant triangle shape deformation.

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